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Acoustic and disturbance energy analysis of a flow with heat communication

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This paper presents a comparative analysis of the budgets of acoustic energy and Myers' second-order 'disturbance energy' in a simple inhomogeneous flow with heat communication. The flow considered is non-diffusive and one-dimensional, with excitation by downstream-travelling acoustic and entropic disturbances. Two forms of heat communication are examined: a case with only steady heat communication and another in which unsteady heat addition cancels the generation of entropy disturbances throughout the inhomogeneous region.

It is shown that significant entropic disturbances are usually generated at low frequency when a flow with steady heat communication is excited either acoustically or entropically. However, for acoustic excitation and regardless of the form of heat communication, entropic disturbances are not created at high frequency, inferring that all source terms create mainly sound in this limit. A general method is therefore proposed for determining an approximate frequency beyond which the generation of entropy disturbances can be ignored, and the disturbance energy flux then approximates the acoustic energy flux. This frequency is shown to depend strongly on the problem under investigation, which is expected to have practical significance when studying sound generation and propagation in combusting flows in particular. Further, sound is shown to be generated by fluid motion experiencing only steady heat communication, which is consistent with the known mechanism of sound generation by the acceleration of density disturbances.

1. Introduction

Since Rayleigh (1896), several forms of acoustic energy conservation equations for various flows have appeared. Candel (1975) reviews most of these acoustic energies in situations where there are no source terms. There are several sources of sound in flows, including turbulence (see Lighthill 1952, 1954) and accelerating density inhomogenities (see Morfey 1971; Howe 1975; Ffwocs Williams & Howe 1975). It is also well known that unsteady heat communication can act as a source or sink of acoustic energy (see Rayleigh 1878). To account for these sources, a number of acoustic energy conservation equations have been developed, of which Morfey (1971) is the most cited. Morfey generalizes the earlier work of Cantrell & Hart (1964) to non-isentropic heat-conducting viscous flows. Morfey's work was later extended by Bloxsidge *et al.* (1988) to include mean and unsteady heat addition in a one-dimensional flow.

While the above references seek a conservation equation for purely acoustic energy, other works have put forward other forms of energy that represent all disturbances

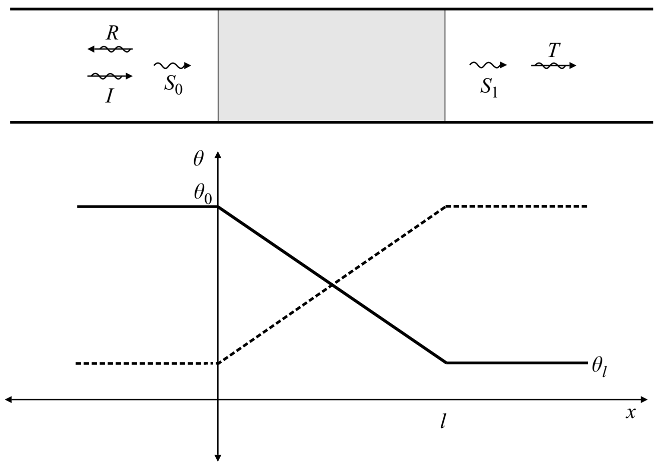

FIGURE 1. Schematic of the problem. Heating, (dotted line) $\bar{\theta}_0 = 600\,\mathrm{K}$, $\bar{\theta}_l = 1200\,\mathrm{K}$ and cooling, (solid line) $\bar{\theta}_0 = 1200\,\mathrm{K}$, $\bar{\theta}_l = 600\,\mathrm{K}$, $l = 1\,\mathrm{m}$.

in a flow. Chu (1965) developed such a corollary by adding the energy carried by entropy disturbances to the acoustic energy in a stationary homentropic flow. Pierce (1989) and Myers (1986) later expressed similar corollaries for homogeneous mean flows. Nicoud & Poinsot (2005) argued that an energy corollary that includes entropy disturbances should replace conventional criteria for the stability of combusting flows based on the conservation of acoustic energy, although this is yet to be successfully demonstrated. Giauque *et al.* (2006) extended the energy corollary of Myers (1991) to full non-equilibrium combustion chemistry. The exact equations were closed on an unsteady laminar flame showing that several terms in the disturbance energy can contribute significantly to the total energetics of the flow. Nonetheless, the complexity of chemical non-equilibrium makes it difficult to understand acoustic and disturbance energy transport in such flows.

Given the same governing equations of fluid motion, any disturbance energy corollary and the acoustic energy equation must be equivalent (see Myers 1991). However, a disturbance energy corollary explicitly includes energies of those disturbances that are not solely acoustic in the energy density and flux terms. It is one aim of this paper to show the relationship between disturbance energy and acoustic energy in flows with heat communication. This will be accomplished by closing the extension of Myer's second-order disturbance energy corollary for one-dimensional non-diffusive flow at chemical equilibrium, presented previously by Giauque *et al.* (2006).

It will be shown that for a given configuration, the difference between the acoustic and disturbance energies depends strongly on the forcing frequency and unsteady heat communication. For acoustic excitation, an 'entropic corner frequency' is therefore proposed, beyond which the generation of entropy disturbances can be ignored and the problem is approximately acoustic. It is argued that this proposed frequency may allow a less problematic analysis of the sources of higher-frequency sound in some inhomogeneous flows. As such, it is intended that this paper will provide some insight into the continued research regarding the unambiguous separation of acoustic, entropic and vortical disturbances in inhomogeneous flows with and without heat communication (for example see Doak 1989; Jenvey 1989).

The problem under investigation is shown in figure 1. The mean flow is one-dimensional and from left to right. The fluid is assumed to be an inviscid non-heat-conducting perfect gas. A region of length l with a linear temperature gradient is connected at either end to homogeneous, semi-infinite regions. When mean flow is present, this mean temperature gradient is sustained by heat communication, i.e. heat addition to or subtraction from the flow. In these homogeneous regions the linear acoustic and entropic disturbances are decoupled and move at the adiabatic sound speed and mean velocity, respectively. Excitation is provided by incident downstream travelling acoustic I or entropic s_0 disturbances. The response of the system can be characterized by reflected R and transmitted T acoustic waves, as well as an outgoing entropic disturbance s_l. Unless stated otherwise, throughout this paper the inlet and exhaust static temperatures and length of the inhomogeneous region are those specified in figure 1.

2. Theoretical and numerical methods

2.1. *Theory*

2.1.1. *Equations of motion*

Consider the one-dimensional Euler equations applied to a calorifically perfect, ideal gas,

$$\frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial x}(\rho u) = 0, \tag{2.1a}$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(p + \rho u^2) = 0, \tag{2.1b}$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\gamma - 1} + \tfrac{1}{2}\rho u^2\right) + \frac{\partial}{\partial x}\left(\left[\frac{\gamma p}{\gamma - 1} + \tfrac{1}{2}\rho u^2\right] u\right) = q, \tag{2.1c}$$

where p (Pa), ρ (kg m^{-3}) and u (m s^{-1}) are, respectively, the static pressure, static density and flow velocity, q (W m^{-3}) is the heat communication per unit volume and γ is the ratio of specific heats ($\gamma = 1.4$ for all presented calculations). Let $\bar{g}$ represent steady quantities satisfying the above equations when all time derivatives are zero, so that any flow property may be expressed as the superposition of its steady and disturbance parts such that $g = \bar{g} + g'$.

2.1.2. *Disturbance energy*

Myers (1991) derives his exact disturbance energy corollary from the conservation equations of mass, momentum and energy as well as the entropy transport equation. He shows that such an energy accommodates a conservative form with source terms. Giauque *et al.* (2006) added species transport and non-equilibrium chemical reaction to Myers' exact corollary and then approximated their exact equations to second-order about a steady mean flow. They stated that such a second-order corollary should be applied with care in combusting flows, mainly since entropic disturbances in particular are often large. Here, for the sake of simplicity, all terms associated with chemical non-equilibrium, viscous and thermal diffusion will be ignored, in which case Myers' energy corollary becomes

$$\frac{\partial E}{\partial t} + \nabla \cdot \boldsymbol{W} = D. \tag{2.2}$$

In (2.2), the second-order disturbance energy density E (J M^{-3}), flux vector $\boldsymbol{W}$ (W m^{-2}) and the second-order source term D (W m^{-3}) are defined, respectively, as

$$E = \frac{p'^2}{2\bar{\rho}\bar{c}^2} + \tfrac{1}{2}\bar{\rho}\cdot\boldsymbol{u}'^2 + \rho'\bar{\boldsymbol{u}}\cdot\boldsymbol{u}' + \frac{\bar{\rho}\bar{\theta}s'^2}{2\bar{c}_p}, \tag{2.3a}$$

$$W = (p' + \bar{\rho}\bar{\boldsymbol{u}}\cdot\boldsymbol{u}')\left(\boldsymbol{u}' + \frac{\rho'}{\bar{\rho}}\bar{\boldsymbol{u}}\right) + \bar{\boldsymbol{m}}\theta's', \tag{2.3b}$$

$$D = \bar{\rho}\bar{\boldsymbol{u}}\cdot(\boldsymbol{\xi}'\times\boldsymbol{u}') + \rho'\boldsymbol{u}'\cdot(\bar{\boldsymbol{\xi}}\times\bar{\boldsymbol{u}}) - s'\boldsymbol{m}'\cdot(\nabla\bar{\theta}) + s'\bar{\boldsymbol{m}}\cdot(\nabla\theta') + \left(\frac{q'\theta'}{\bar{\theta}} - \frac{\bar{q}\theta'^2}{\bar{\theta}^2}\right), \tag{2.3c}$$

where θ (K) is temperature, $\boldsymbol{u}$ (m s^{-1}) and $\boldsymbol{\xi}$ (s^{-1}) are the velocity and vorticity, $\boldsymbol{m} = \rho\boldsymbol{u}$ is the mass flux and c_p (J kg^{-1}K^{-1}) is the gas specific heat at constant pressure.

In the cases of isentropic and homentropic flows, (2.2) to (2.3c) simplify to existing acoustic energy conservation equations. For homentropic flow, the energy density and flux terms become those defined by Cantrell & Hart (1964) for acoustic propagation in a non-stationary medium. In non-isentropic flows without heat communication, Myers argued that (2.2) to (2.3c) are equivalent to the equations presented by Morfey (1971). In the source term D, the first two terms represent the generation of disturbance energy by unsteady vortical motion. The next two terms represent the effect of entropy disturbances and their interaction with temperature, density and velocity disturbances. The final two terms describe the effect of steady and unsteady heat communication interacting with temperature disturbances, with $q'\theta'/\bar{\theta}$ analogous to the source term in the so-called 'Rayleigh criterion' (see Rayleigh 1878; Nicoud & Poinsot 2005).

2.2. *Numerical solver*

The present work solves (2.1a)–(2.1c) in conservation form by using the dispersion-relation-preserving (DRP) scheme of Tam & Webb (1993). The specific DRP scheme chosen uses an optimized four-level time-marching scheme and seven-point stencil for spatial differentiation. The choice of such a scheme ensures that the computed waves are a good approximation of the exact Euler equations. Non-reflecting boundary conditions are implemented to ensure that the numerical domain approximates an infinite domain. The exact boundary conditions follow the same formulation given by Poinsot & Lele (1992) to ensure that the incoming waves at each boundary are always zero. An exception to this is in the implementation of the system excitation. The system can be forced by an incoming downstream-travelling pressure or entropy wave at the inlet. The amplitude of the system excitation is small enough to ensure that the system is always linear. Numerical damping is employed to remove non-physical high-frequency waves from the solution. This is a modified version of the scheme of Tam & Shen (1993), and incorporates damping in regions of entropy discontinuity. All simulations are run with a Courant–Friedrichs–Lewy number of 0.1. The number of grid points in the inhomogeneous region used in each simulation is 601. Simulations are run for a sufficiently long time to ensure that no transients are present in the final results. This solver has been validated on several problems, such as those presented in Moase, Brear & Manzie (2007), as well as on the results presented in this paper.

3. Results and discussion

3.1. *Acoustic energy analysis*

This section examines acoustic wave reflection and transmission by the one-dimensional system shown in figure 1. The acoustic energy reflection and transmission

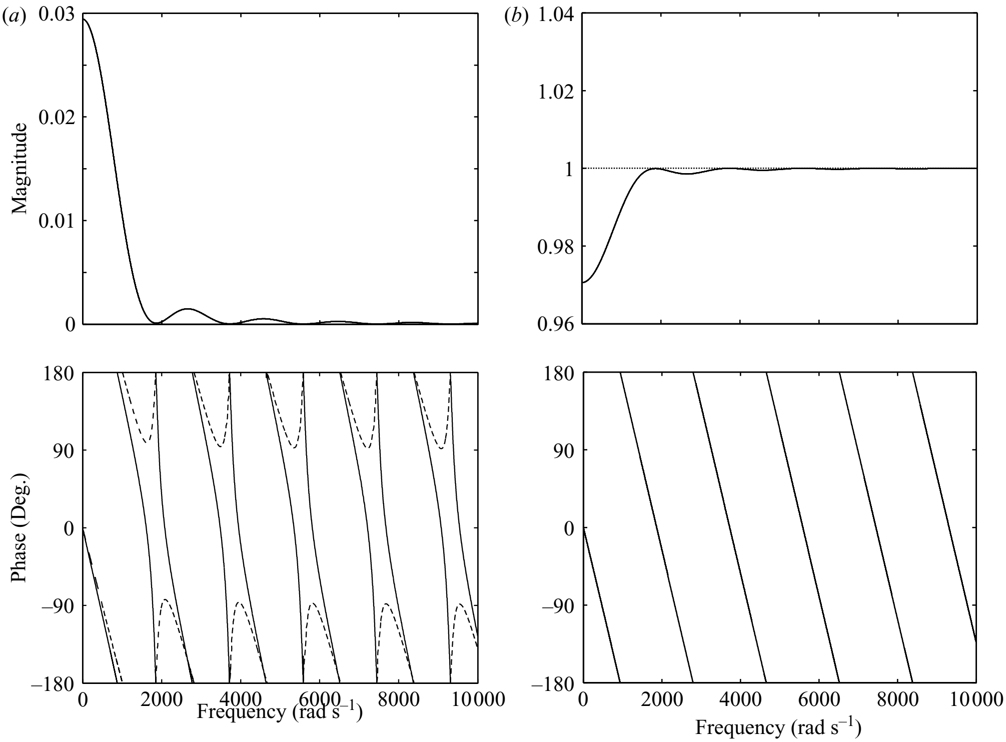

FIGURE 2. Analytic frequency response of the (*a*) acoustic energy reflection and (*b*) transmission coefficients. $\bar{M} = 0$, cooling (solid line) and heating (dashed line).

coefficients are defined as the ratio of the reflected or transmitted acoustic energy flux to the incident energy flux. It should first be clarified that the concepts of acoustic reflection and transmission used in this paper are slightly different to those in classical acoustics. This is due to the possible generation/dissipation of sound in the heat communicating flows which will be further investigated in the present section.

3.1.1. *Zero mean flow*

Figure 1 is first considered with zero mean flow and no unsteadiness in heat communication. There is also no steady heat communication, since the temperature profile is an initial condition in this non-diffusive flow. Convection of entropy waves cannot occur and transport of energy in and out of the domain must be described by the classical acoustic energy flux (see Candel 1975). To satisfy the first law of thermodynamics, the time-averaged classical acoustic energy fluxes must be conserved, resulting in (see Dowling & Ffowcs Williams 1983)

$$\frac{1}{\bar{\rho}_0\bar{c}_0}|R|^2 + \frac{1}{\bar{\rho}_l\bar{c}_l}|T|^2 = \frac{1}{\bar{\rho}_0\bar{c}_0}|I|^2, \tag{3.1}$$

where the wave amplitudes are $R = (p'_0 + u'_0/\bar{\rho}_0\bar{c}_0)/2$, $I = (p'_0 - u'_0/\bar{\rho}_0\bar{c}_0)/2$ and $T = (p'_l + u'_l/\bar{\rho}_l\bar{c}_l)/2$. Thus, the reflected and transmitted energy coefficients are defined, respectively, as $|R|^2/|I|^2$ and $(\bar{\rho}_0\bar{c}_0/\bar{\rho}_l\bar{c}_l)|T|^2/|I|^2$ where c is the adiabatic sound speed.

This purely acoustic problem can be solved analytically by extending the results in Sujith, Waldherr & Zinn (1995). As presented in Appendix A, analytic expressions can be derived for the reflected and transmitted acoustic waves and the associated classical acoustic energy reflection and transmission coefficients. Figure 2 shows the results of that exact analysis for both heated and cooled flows. According to this figure, in a stationary medium, a small amount of the incident energy is reflected at low frequencies while all of the energy is transmitted as $\omega \rightarrow \infty$. The sum of the reflected and transmitted energy at a given frequency is always equal to 1, as (3.1) requires. The results of the exact analysis presented here also agree qualitatively with the numerically computed acoustic reflection and transmission coefficients for a similar configuration presented in Candel, Defillipi & Launay (1980*a*, *b*).

3.1.2. *Non-zero mean flow*

The inclusion of non-zero mean flow allows the convection of entropy disturbances and also requires continuous heat addition or extraction for maintaining the mean temperature gradient. This is incorporated in the problem by assuming that the flow has heat communication from external means, which is a reasonable approximation for a reacting flow in the equilibrium chemistry limit, as well as incident or emitted radiation, for example. The numerical solver detailed in § 2.2 is used to aid the analysis. Here only two forms of unsteadiness in heat communication are considered: steady heat communication and a specific type of unsteady heat communication which produces no entropy. The latter is accomplished by considering the linearized entropy transport equation (see Dowling 1995)

$$\frac{\partial s'}{\partial t} + \bar{u}\frac{\partial s'}{\partial x} = \frac{\bar{q}R}{\bar{p}}\left(\frac{q'}{\bar{q}} - \frac{u'}{\bar{u}} - \frac{p'}{\bar{p}}\right). \tag{3.2}$$

Thus, heat communication will produce no entropy disturbances in the domain if

$$\frac{q'(x,t)}{\bar{q}(x)} = \frac{u'(x,t)}{\bar{u}(x)} + \frac{p'(x,t)}{\bar{p}(x)}. \tag{3.3}$$

Suppression of entropy disturbances in this way makes the problem purely acoustic, as (2.2) to (2.3*c*) show. This is a degenerate case and exceedingly unlikely to occur in a combusting flow. However, it is the one instance in which the source term in (2.2) for the present one-dimensional problem becomes exactly purely acoustic. As will be discussed later, this is of importance for the present study.

3.1.3. *Acoustic excitation*

The mean acoustic energy flux defined by Cantrell & Hart (1964) within the homogeneous regions in figure 1 is the mean of

$$p'u' + \bar{\rho}\bar{u}u'^2 + \frac{\bar{u}^2 p'u'}{\bar{c}^2} + \frac{\bar{u}p'^2}{\bar{\rho}\bar{c}^2}. \tag{3.4}$$

Substituting the earlier definitions of R, I and T into (3.4) the acoustic energy reflection and transmission coefficients can be defined as the ratio of the reflected and transmitted acoustic energy flux to the incident acoustic energy flux, and are, respectively,

$$\frac{(1-\bar{M}_0)^2}{(1+\bar{M}_0)^2}\left|\frac{R}{I}\right|^2, \quad \frac{\bar{\rho}_0\bar{c}_0}{\bar{\rho}_l\bar{c}_l}\frac{(1+\bar{M}_l)^2}{(1+\bar{M}_0)^2}\left|\frac{T}{I}\right|^2, \tag{3.5a, b}$$

where M is the flow Mach number.

The problem of reflection and transmission of acoustic energy with non-zero finite mean flow can be solved analytically for a compact inhomogeneous region. The results of this analysis for both cases of heat unsteadiness are shown in figure 3,

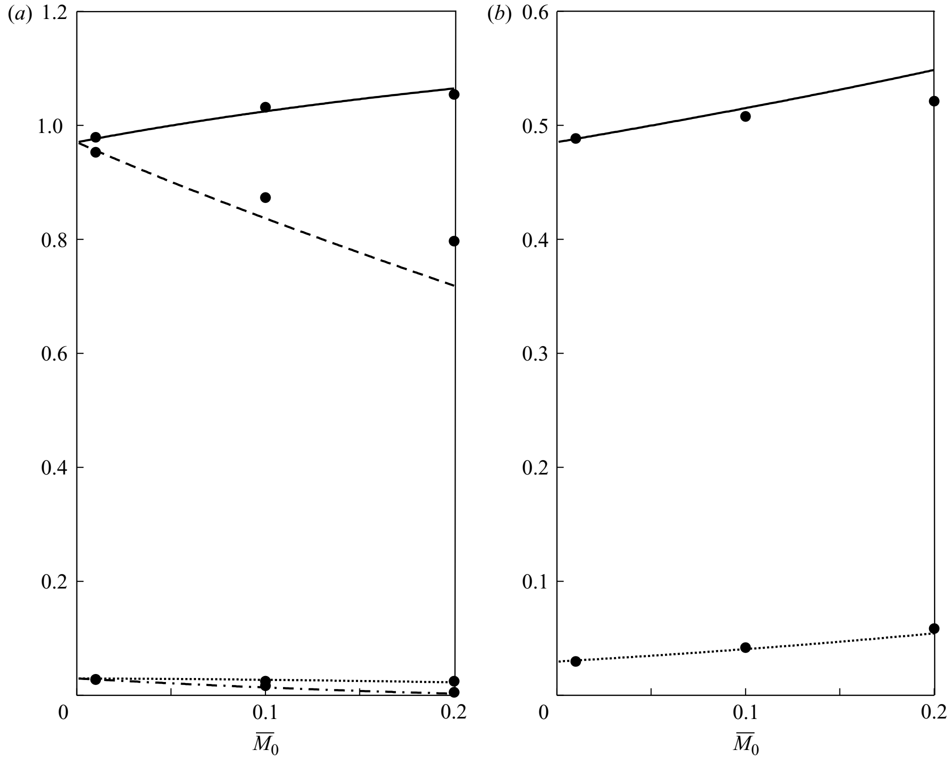

FIGURE 3. Amplitudes of the acoustic energy reflection and transmission coefficients for a compact region with mean flow, excited acoustically and with (*a*) $q' = 0$ and (*b*) $s'_{gen} = 0$; analytical transmission coefficients: cooling (solid lines) and heating (dashed line), analytical reflection coefficients: cooling (dotted line) and heating (dash-dotted line), simulation (•).

with the full expressions given in Appendix B. Figure 3 also shows the reflection and transmission coefficients evaluated numerically for the lowest frequency of excitation studied, $\omega = 200$ rad s^{-1}. The agreement between these two sets of results serve in part to validate the numerical solver. The small disagreement can be attributed primarily to the non-zero frequency used in the numerical simulation (as the excitation frequency tends to zero, the solver takes infinitely long to reach the steady state).

Now consider the frequency response of the coefficients defined in (3.5). Figure 4 shows results with the unsteady heat communication set to zero. This means that the so-called 'Rayleigh term' $q'\theta'/\bar{\theta}$ plays no part. The transmission coefficient exceeds unity, indicating that acoustic energy is now generated within the domain. This sound generation is due to the interaction of steady heat communication with acoustic and entropic disturbances, as explained later in this paper. In acoustic analyses of flows with heat communication, the sound generation by steady heat communication is often ignored and sound production or attenuation is incorrectly attributed to only the unsteadiness in heat communication.

Figure 5 shows this response in the case of zero entropy generation for the two non-zero Mach numbers considered in figure 4. Clearly, introduction of a mean flow and unsteady heat communication have only slightly modified the acoustic energy reflection coefficient. However, the transmission coefficient has dropped to a value of about half that for zero unsteady heat communication, indicating that almost half

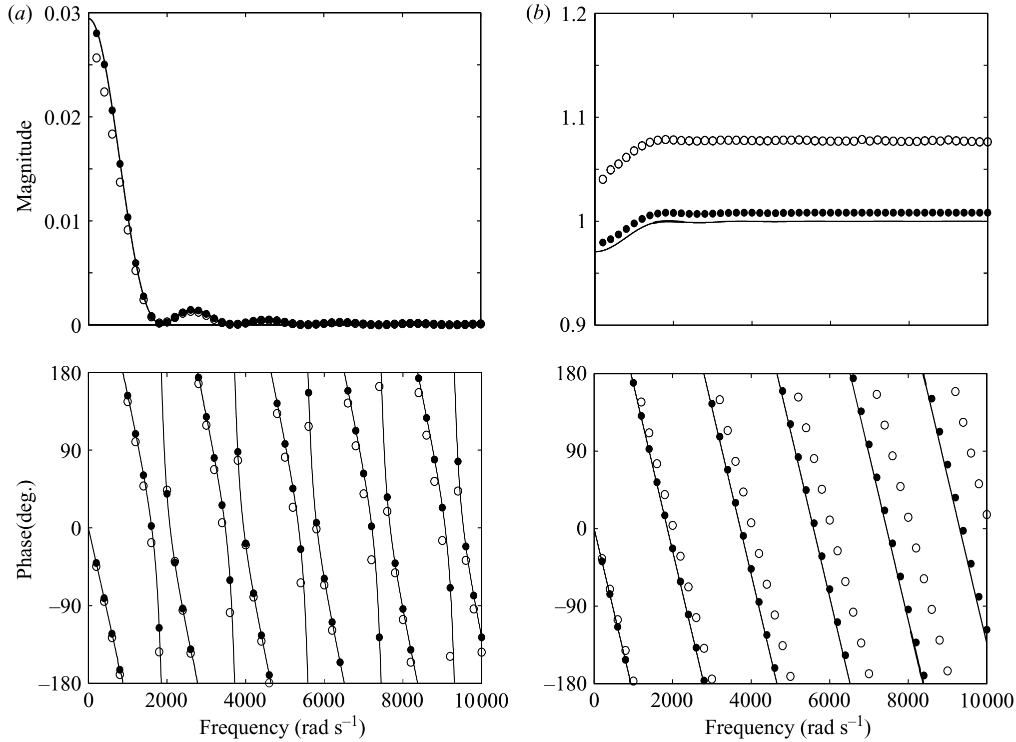

FIGURE 4. Frequency response of the acoustic energy (*a*) reflection and (*b*) transmission coefficients, cooling case and $q'=0$; $\bar{M}=0$ analytical (solid lines); $\bar{M}_0=0.01$, simulation (•); $\bar{M}_0=0.1$, simulation (○).

of the incident acoustic energy has been dissipated. This is due to the acoustic field interacting with the unsteady heat communication.

3.1.4. *Entropic excitation*

In this section, the excitation is entirely due to convective entropy disturbances. The conversion of entropic disturbance energy to acoustic energy takes place owing to the interaction of the entropy wave with the accelerating or decelerating mean flow (see Morfey 1971; Ffwocs Williams & Howe 1975), and also involves interaction between the generated acoustic wave and the steady heat communication. These two mechanisms are similar since steady heat addition is proportional to the steady temperature gradient. The acceleration of a non-diffusive work-free flow also implies gradients in the mean static temperature field since the stagnation temperature is constant, in turn implying a non-zero source term $s'\boldsymbol{m}'\cdot(\nabla\bar{\theta})$ in (2.3*c*).

In terms of the wave amplitudes in figure 1, the time average of the fluxes of disturbance energy in the homogeneous regions upstream and downstream are, respectively,

$$\frac{1}{2}\left[|s_0'|^2\frac{\bar{\rho}_0\bar{M}_0\bar{c}_0^3}{c_p^2(\gamma-1)}+|s_0'||R|\frac{\bar{c}_0\bar{M}_0^2}{c_p}-\frac{(1-\bar{M}_0)^2}{\bar{\rho}_0\bar{c}_0}|R|^2\right], \tag{3.6a}$$

$$\frac{1}{2}\left[|s_l'|^2\frac{\bar{\rho}_l\bar{M}_l\bar{c}_l^3}{c_p^2(\gamma-1)}-|s_l'||T|\frac{\bar{c}_l\bar{M}_l^2}{c_p}+\frac{(1-\bar{M}_l)^2}{\bar{\rho}_l\bar{c}_l}|T|^2\right]. \tag{3.6b}$$

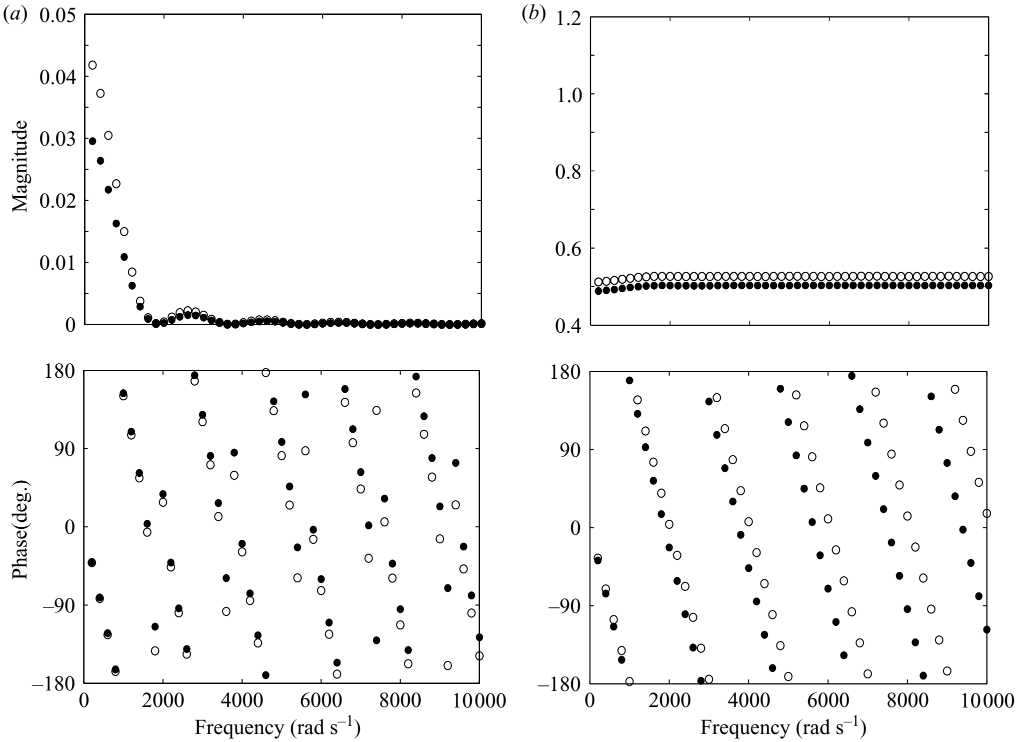

FIGURE 5. Frequency response of the acoustic energy (*a*) reflection and (*b*) transmission coefficients, cooling case and $s'_{gen} = 0$; $\bar{M}_0 = 0.01$, simulation (●); $\bar{M}_0 = 0.1$, simulation (○).

The ratio of the reflected and transmitted acoustic energy fluxes to the incident entropic energy can be written as, respectively,

$$\frac{c_p^2(\gamma-1)(1-\bar{M}_0)^2}{\bar{\rho}_0^2\bar{M}_0\bar{c}_0^4}\left|\frac{R}{s'_0}\right|^2, \quad \frac{c_p^2(\gamma-1)(1+\bar{M}_l)^2}{\bar{\rho}_0\bar{\rho}_l\bar{M}_0\bar{c}_0^3\bar{c}_l}\left|\frac{T}{s'_0}\right|^2. \qquad (3.7a,b)$$

It should be noted that the above defined reflection and transmission coefficients are valid for subsonic flows only. It is also worth noting that the ratio of the total acoustic energy generated to the incident entropic energy is simply the sum of these two coefficients.

A compact analysis for entropic excitation is presented in Appendix B. Figure 6 compares the numerical reflection and transmission coefficients with the compact analysis for a cooling case. As can be seen, the two sets of results agree well, with most of the disagreement again due to the non-zero frequency ($\omega = 200\,\mathrm{rad\,s^{-1}}$) used in the numerical simulations. This figure also shows a strong Mach-number dependence of the coefficients, with singular results for sonic flow.

The production of acoustic energy by entropic forcing is small at low Mach numbers, so $\bar{M} = 0.5$, is now considered. Figure 7 shows the frequency response of the reflection and transmission coefficients for the two types of heat unsteadiness discussed earlier. Most of the sound production occurs in the low-frequency range, whereas at high frequencies the conversion is negligible.

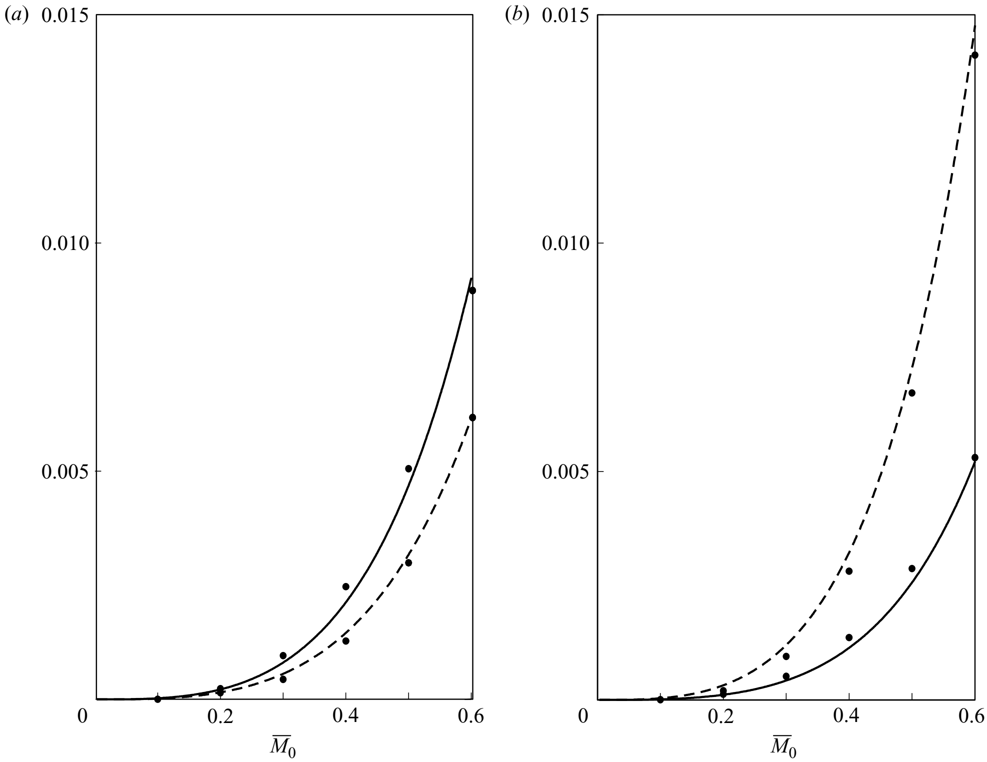

FIGURE 6. Amplitudes of the reflection and transmission coefficients defined in (3.7) for a compact region with mean flow excited entropically and (*a*) $q' = 0$ and (*b*) $s'_{gen} = 0$, cooling, analytical reflection coefficient (dashed line) and transmission coefficient (solid line); simulation (•).

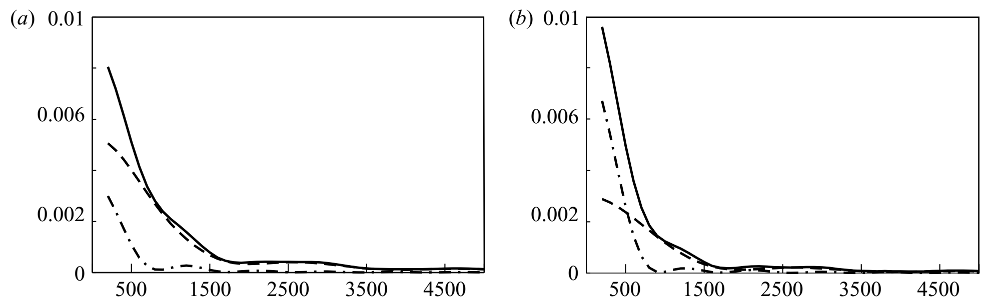

FIGURE 7. Amplitude of the reflection and transmission coefficients defined in (3.7) for entropically excited flow and (*a*) $q' = 0$ and (*b*) $s'_{gen} = 0$, cooling , $\bar{M}_0 = 0.5$, reflection (dash-dotted line), transmission (dashed line), reflection plus transmission (solid line).

3.2. *Disturbance energy analysis*

The balance of disturbance energy is given by (2.2). This equation is closed on the present problem and it is shown how mean flow effects appear in the overall disturbance energy budget of the system. In doing so, (2.2) is first integrated over the entire inhomogeneous region and then averaged over several periods of excitation.

After applying the divergence theorem, this results in

$$\int_0^l \overline{\frac{\partial E}{\partial t}}\,\mathrm{d}x + [\overline{W}]_0^l = \int_0^l \overline{D}\,\mathrm{d}x. \tag{3.8}$$

The second term on the left-hand side of (3.8) is the time average of the net flux of disturbance energy out of the region. Expanding this term using its components in (2.3c) reveals

$$[\overline{W}]_0^l = f_1 + f_2 + f_3 + f_4 + f_5, \tag{3.9}$$

where

$$f_1 = [\overline{p'u'}]_0^l, \quad f_2 = [\overline{p'\rho'\bar{u}/\bar{\rho}}]_0^l, \quad f_3 = [\overline{\bar{\rho}\bar{u}u'^2}]_0^l, \tag{3.10a,c}$$

$$f_4 = [\overline{\bar{u}^2\rho'u'}]_0^l, \quad f_5 = [\overline{\bar{m}\theta's'}]_0^l. \tag{3.10d,e}$$

Similarly, for the source term we have

$$\int_0^l \overline{D}\,\mathrm{d}x = d_1 + d_2 + d_3 + d_4, \tag{3.11}$$

where

$$d_1 = \int_0^l \left[\overline{-s'm'}\left(\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}x}\right)\right]\mathrm{d}x, \quad d_2 = \int_0^l \overline{\left[s'\bar{m}\left(\frac{\partial\theta'}{\partial x}\right)\right]}\mathrm{d}x, \tag{3.12a,b}$$

$$d_3 = \int_0^l \left(\frac{\overline{q'\theta'}}{\bar{\theta}}\right)\mathrm{d}x, \quad d_4 = \int_0^l \left(\frac{\bar{q}\overline{\theta'^2}}{\bar{\theta}^2}\right)\mathrm{d}x. \tag{3.12c,d}$$

The above equations are applied to the flows analysed previously.

3.2.1. *Zero mean flow*

In this case for zero unsteadiness in heat communication, all source terms must be zero otherwise the initial temperature distribution would be non-stationary and the requirement of a steady mean flow would be violated. The source terms d_2, d_3 and d_4 in (3.12) are clearly zero. The source term d_1 is also zero, for less obvious reasons.

For zero mean flow and without unsteady heat communication, the linearized entropy transport equation is

$$\frac{\partial s'}{\partial t} = -\boldsymbol{u}'\cdot\nabla\bar{s}. \tag{3.13}$$

Thus, a propagating harmonic acoustic wave causes a harmonic variation in entropy at a point in space, but no entropy generation since the substantial derivative of s' is always zero. For harmonic excitation, s' must therefore be orthogonal to u'. The total source term can therefore be written as

$$D = -s'\bar{\rho}\boldsymbol{u}'\cdot\nabla\bar{\theta}. \tag{3.14}$$

Since u' and s' are orthogonal, the time average of the (3.14) is zero and the entropy oscillations in (3.13) do not, on average, create any disturbance energy. Since entropy disturbances cannot enter or leave the stationary homogeneous regions, it follows that the unsteady energy fluxes are entirely acoustic. The acoustic and disturbance energies are therefore equivalent, as required by the first law of thermodynamics and shown earlier in § 3.1.1.

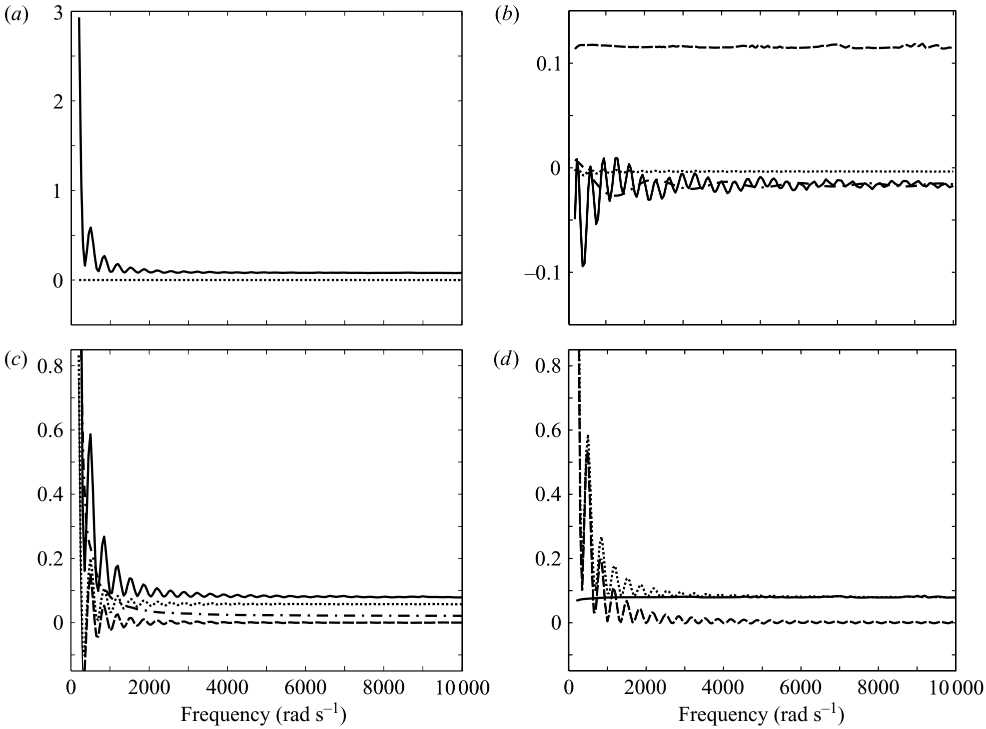

FIGURE 8. Closure of the disturbance energy corollary for acoustically excited flow, cooling and $q' = 0$, (a) overall balance, (solid line); $\int_0^l \overline{(\partial E/\partial t)} \mathrm{d}x$ (dotted line), (b) source terms d_1 (dashed line), d_2 (dotted line), d_4 (dash-dotted line), total source term (solid line); (c) flux terms f_1 (dashed line), f_2 (solid line), f_3 (dash-dotted line), f_4 (dotted line); d) flux terms f_5 (dashed line), total flux (dotted line), net flux of acoustic energy (solid line).

3.2.2. *Acoustic excitation of non-zero mean flow*

Figure 8 shows the results of the disturbance energy analysis for the acoustically excited flow with the mean flow being cooled. All quantities in this figure have been non-dimensionalized by the incident acoustic energy flux. Figure 8(a) presents the closure of (2.2) with zero heat unsteadiness and $\bar{M} = 0.1$ (as studied earlier in figures 4 and 5). As expected the frequency response of the flux and source terms in (3.8) are equal, and the first term on the left-hand side of (3.8) is zero. There is a strong frequency dependence below 2000 rad s^{-1} in this test case. All terms become less frequency dependent at higher frequencies. The occurrence of these peaks and troughs stems from the existence of the acoustic and convective length scales in the problem. Excitation of any of the frequencies associated with these length scales may cause a peak in the budget of a given term of the disturbance energy equation, depending on the dominance of either acoustic or convective phenomena in that particular term.

The contribution of each source term in (2.3c) to the total generation of disturbance energy is also shown in figure 8(b). Since there is no unsteadiness in the heat absorption, only three of the source terms (d_1, d_2 and d_4) can have an effect. The frequency dependence below 2000 (rad s^{-1}) and frequency independence at high frequency is once again observed. In general, the disturbance energy is a combination of acoustic and entropic terms. However, the net flux of one entropic term, f_5, shows

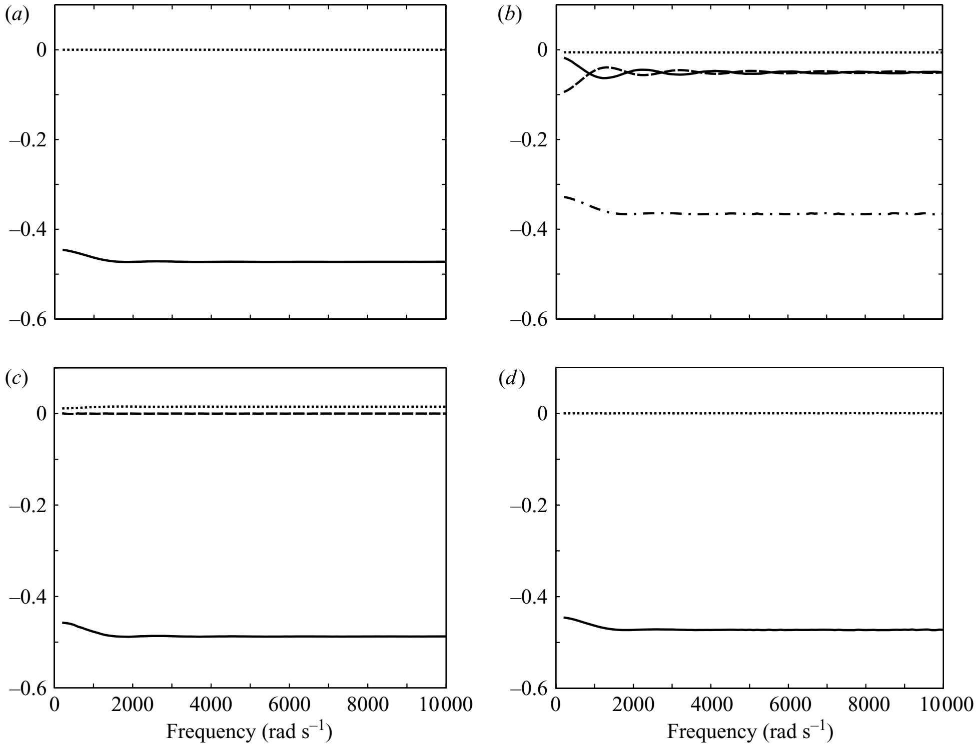

FIGURE 9. Closure of the disturbance energy corollary for acoustically excited flow, cooling and $s'_{gen}=0$, (*a*) overall balance, (solid line); $\int_0^l (\overline{\partial E/\partial t})\mathrm{d}x$ (dotted line); (*b*) source terms, d_1 and d_2 (dashed line), d_3 (solid line), d_4 (dotted line); (*c*) flux terms, f_1 (dash-dotted line), f_2 (solid line), f_3 (dashed line), f_4 (dotted line); (*d*) flux terms, f_5 (dotted line), total flux (solid line).

that entropy generation in the region tends to zero at higher frequencies. Hence, the contribution of entropy disturbances to the flux of disturbance energy becomes negligible as frequency increases and, as a result, the flux of disturbance energy approaches Cantrell & Hart's flux of acoustic energy. This behaviour is discussed further in § 3.3.1.

Unsurprisingly, suppression of entropy production by unsteady heat communication by using (3.14) has a major effect (figure 9). As is required, the flux and source terms are again the same. Most notably, there is less frequency dependence compared to that in the zero heat unsteadiness case. This can be attributed to the elimination of convective phenomena owing to suppression of entropy disturbances. The negative terms are due to dissipation of disturbance energy. Since there are no entropy disturbances in the domain, the disturbance energy density and flux terms in this very particular case equal those defined by Cantrell & Hart (1964). The only non-zero source terms in this case are those due to mean and fluctuating heat communication.

Although not shown here, very similar dynamic behaviour can be observed in the case of heating the flow. Of course, the overall gain or loss of disturbance energy is the opposite of the results presented here for the cooled mean flow.

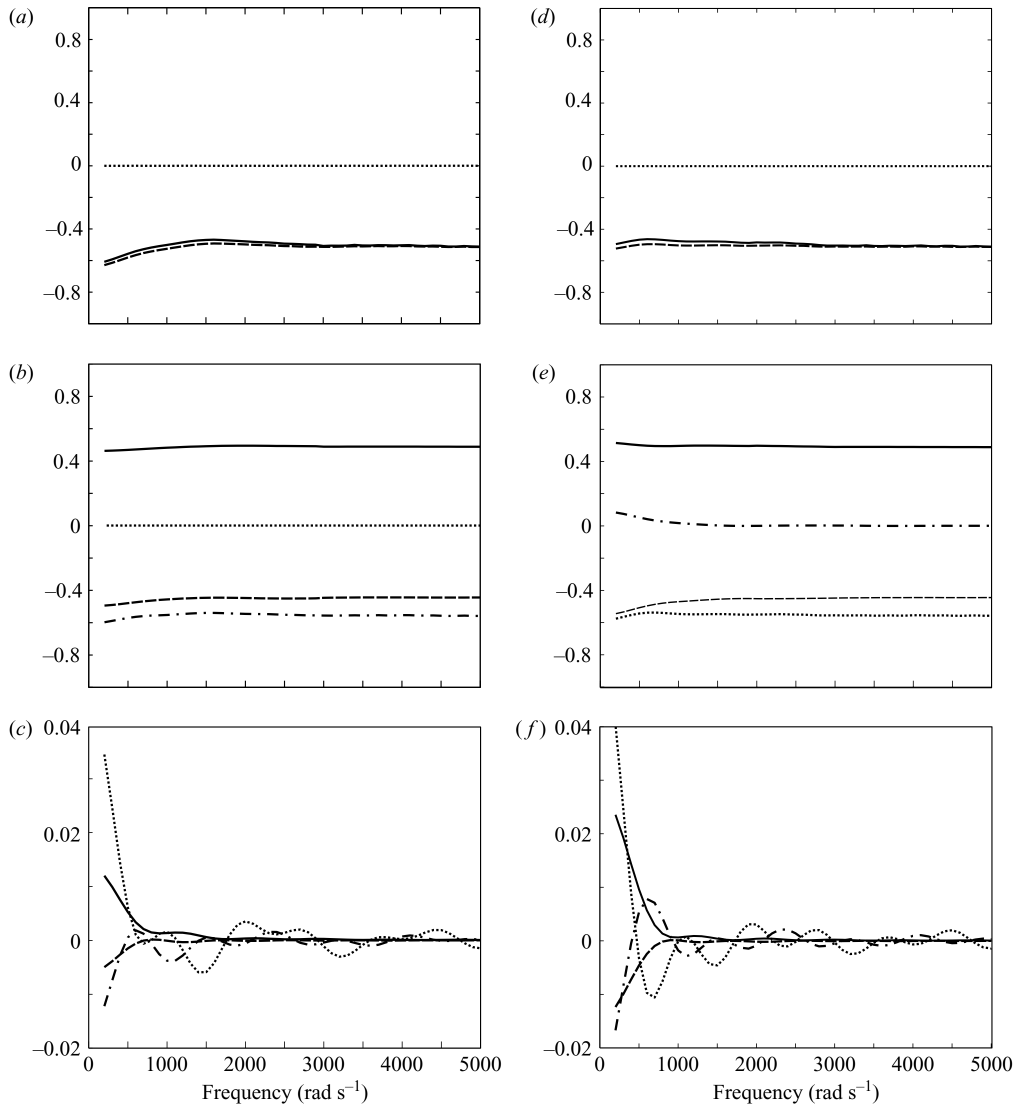

FIGURE 10. Closure of the disturbance energy corollary for entropically excited flow, cooling and (a, b and c) $q'=0$ and (d, e and f) $s'_{gen}=0$; (a, d) overall balance, right-hand side and left-hand side of (3.8) (solid line and dashed line); $\int_0^l(\overline{\partial E/\partial t})\mathrm{d}x$ (dotted line); (b, e) source terms, d_1 (dashed line), d_2 (dash-dotted line), d_3 (dotted line), d_4 (solid line); (c, f) flux terms f_1 (solid line) f_2 (dotted line), f_3 (dash-dotted line), f_4 (dashed line).

3.2.3. *Entropic excitation of non-zero mean flow*

The total energetics of the system under entropic excitation are shown in figure 10. All values have been non-dimensionalized by the energy flux of the incident entropy wave, which is the first term in (3.6a). While d_1 and d_2 act as a sink of disturbance energy, d_4 adds energy to the system at nearly the same magnitude. The result is an overall sink which dissipates more than half of the incident disturbance energy flux. Unsurprisingly, the flux of disturbance energy is mostly due to f_5 which represents energy carried by entropy disturbances. This is supported by the plots of f_1 to f_4 in figure 10(c) which shows that these fluxes are all at least an order of magnitude less than the net flux in the overall balance on the same figure.

Figure 10(d–f) shows that suppression of entropy disturbances in the inhomogeneous region only modifies the results slightly. This is predictable since in the previous case the acoustics are relatively weak and generate only small amounts of entropy.

It is initially surprising to observe a loss in disturbance energy when there is no entropy generation, since the incident entropy wave remains unchanged and a small amount of sound is generated. However, it should be noted that the drop in the mean temperature while entropy is kept constant reduces the amplitude of temperature disturbances and results in the reduction of f_5. Thus, in the case of entropic excitation, the drop in the disturbance energy in both types of heat unsteadiness is almost independent of the effect of entropy and influenced mainly by the mean temperature change which is itself a consequence of mean heat communication.

3.3. *Asymptotic behaviour of entropy generation*

Simple scaling arguments require that the inhomogeneous region becomes homogeneous for entropic disturbances in the high-frequency limit. An entropic disturbance of a given frequency always views the region as more homogeneous, than does an acoustic disturbance of the same frequency because of their relative convective and acoustic length scales. This suggests the existence of a ‘corner’ frequency above which the generation of entropic disturbances becomes small. The following analysis in this section ultimately leads to a method for determining this frequency.

3.3.1. *Entropy generation by acoustic or entropic excitation*

It is assumed that all disturbances are harmonic so that

$$p' = P(x)\exp(\mathrm{i}\omega t), \quad u' = U(x)\exp(\mathrm{i}\omega t), \tag{3.15a, b}$$

$$q' = Q(x)\exp(\mathrm{i}\omega t), \quad s' = S(x)\exp(\mathrm{i}\omega t), \tag{3.15c, d}$$

where ω is the excitation frequency. Substituting these into the entropy transport equation (3.2) and rearranging gives

$$\bar{u}\frac{\mathrm{d}S}{\mathrm{d}x} = \frac{\bar{q}R}{\bar{p}}\left(\frac{Q}{\bar{q}} - \frac{U}{\bar{u}} - \frac{P}{\bar{p}}\right) - \mathrm{i}\omega S. \tag{3.16}$$

When $\omega = 0$, the last term in the right-hand side vanishes and the remaining terms can be integrated to give the change in the amplitude of entropy disturbance over the inhomogeneous region,

$$S_l - S_0 = \int_0^l \frac{\mathrm{d}S}{\mathrm{d}x}\,\mathrm{d}x = \int_0^l \frac{\bar{q}R}{\bar{p}\bar{u}}\left(\frac{Q}{\bar{q}} - \frac{U}{\bar{u}} - \frac{P}{\bar{p}}\right)\mathrm{d}x. \tag{3.17}$$

Note that since $\omega = 0$, $s' = S$. The right-hand side of (3.17) is finite and depends on both the mean and fluctuating properties of the flow.

As the frequency tends to infinity, the second term on the right-hand side of (3.16) dominates the first term and the entropy transport equation reduces to

$$\bar{u}\frac{\mathrm{d}S}{\mathrm{d}x} = -\mathrm{i}\omega S, \tag{3.18}$$

which is simply a rearrangement of a homogeneous transport equation. Thus, if no entropy enters the inhomogeneous region, no entropy appears.

It has therefore been shown that the linearized transport of entropy within any inhomogeneous region is strongly frequency dependent. It varies from a finite value at low frequencies to zero as $\omega \to \infty$. Importantly, these results are general for any form of excitation (acoustic or entropic) and heat communication (steady or unsteady), provided that the disturbances remain linear and heat communication is independent of entropy disturbances.

3.3.2. *An entropic 'corner' frequency*

In this section, we seek a frequency at which the generation of entropy disturbances by acoustic forcing is small enough that it can be ignored. This results in the disturbance energy flux being approximately equal to the acoustic energy flux for higher-frequencies. Starting from the flux of disturbance energy defined in (2.3*b*) and using the following relations for linearized entropy disturbances

$$\frac{\theta'}{\bar{\theta}} = \frac{\gamma - 1}{\gamma}\frac{p'}{\bar{p}} + \frac{s'}{c_p}, \quad \frac{\rho'}{\bar{\rho}} = \frac{1}{\gamma}\frac{p'}{\bar{p}} - \frac{s'}{c_p}, \tag{3.19a,b}$$

the disturbance energy flux can be written in terms of p', u' and s'. For the homogeneous downstream region, there is no reflection of the acoustic wave so $p' = 1/(\bar{\rho}_l \bar{c}_l)u'$. Thus, the disturbance energy in the downstream homogeneous region can be expressed in terms of only u' and s',

$$W_l = W_{ac} - \frac{\bar{\rho}_l \bar{c}_l^2 \bar{M}_l^2}{c_p} u' s' + \frac{\bar{\rho}_l \bar{c}_l \bar{\theta}_l \bar{M}_l}{c_p} s'^2, \tag{3.20}$$

where

$$W_{ac} = (1 + \bar{M}_l)(\bar{\rho}_l \bar{c}_l)^2 \left[\frac{1}{\bar{\rho}_l \bar{c}_l} + \frac{1}{\gamma}\frac{\bar{c}_l \bar{M}_l}{\bar{p}_l}\right] u'^2, \tag{3.21}$$

is the flux of acoustic energy. Dividing (3.20) by W_{ac} gives

$$\frac{W_l}{W_{ac}} = 1 + \epsilon, \tag{3.22}$$

where

$$\epsilon = \frac{\bar{\theta}_l \bar{M}_l / c_p}{(1 + \bar{M}_l)\left[1 + (1/\gamma)\bar{\rho}_l \bar{c}_l^2 \bar{M}_l / \bar{p}_l\right]} \left(\frac{s'}{u'}\right)^2 - \frac{\bar{c}_l \bar{M}_l^2 / c_p}{(1 + \bar{M}_l)\left[1 + (1/\gamma)\bar{\rho}_l \bar{c}_l^2 \bar{M}_l / \bar{p}_l\right]} \left(\frac{s'}{u'}\right). \tag{3.23}$$

We define the corner frequency ω^* as the forcing frequency at which (3.22) results in a sufficiently small value of ϵ. Equation (3.23) determines the value of s'/u' in the downstream homogeneous region for any value of ϵ.

The linearized entropy transport equation (3.2) can be scaled as follows

$$s'\omega^* + \bar{u}\frac{s'}{\lambda_c} \simeq \frac{\bar{q}R}{\bar{p}}\left(\frac{q'}{\bar{q}} - \frac{p'}{\bar{p}} - \frac{u'}{\bar{u}}\right), \tag{3.24}$$

in which the temporal and spatial derivatives, $\partial(\ )/\partial t$ and $\partial(\ )/\partial x$, have been approximated by $(\ )\omega^*$ and $(\ )/\lambda_c$, respectively, where λ_c is the convective wave length defined as $\lambda_c = \bar{u}/\omega^*$ and ω^* is the corner frequency. Thus, as the flow approaches the end of the inhomogeneous region, $x = l$, (3.24) can be rearranged to give

$$2\left(\frac{s_l'}{u_l'}\right)\omega^* \simeq \frac{\bar{q}_l R}{\bar{p}_l}\left[\frac{1}{\bar{q}_l}\left(\frac{q_l'}{u_l'}\right) - \frac{1}{\bar{p}_l}\left(\frac{p_l'}{u_l'}\right) - \frac{1}{\bar{u}_l}\right]. \tag{3.25}$$

ϵ	0.1	0.01	0.001
ω^*(theory)(rad s^{-1})	1698.6	5148.9	14783.7
ω^*(simulation)(rad s^{-1})	1921.5	5200	14 888
error	11.6 %	0.99 %	0.7 %

TABLE 1. Entropic corner frequencies for the cooling case $\bar{q}' = 0$, ($\theta_0 = 1200$ K, $\theta_l = 600$ K, $\bar{M}_0 = 0.1$).

In this equation, s'_l/u'_l is equal to s'/u' in the downstream homogeneous region given by (3.23). The transfer function q'_l/u'_l has arbitrary form in real flows.

Finally, p'_l/u'_l in (3.25) is the acoustic impedance at the end of the inhomogeneous region and is found as follows. Transport of momentum for the present one-dimensional flow can be written as

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0. \tag{3.26}$$

Incorporating the thermodynamic relation (3.19) and the linearized entropy transport equation (3.2), the linearization of the momentum equation gives

$$\bar{\rho}\frac{\partial u'}{\partial t} + \bar{\rho}\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}u' + \frac{1}{\bar{c}^2}\bar{u}\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}p' - \frac{\bar{\rho}\bar{u}}{c_p}\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}s' + \bar{\rho}\bar{u}\frac{\partial u'}{\partial x} + \frac{\partial p'}{\partial x} = 0. \tag{3.27}$$

This equation can be scaled similarly to (3.24) with the spatial derivatives of pressure and velocity disturbances now approximated by ( )$/\lambda_{ac}$ where $\lambda_{ac} = \bar{c}/\omega^*$ is the acoustic wave length. The resulting equation can be rearranged in terms of (p'_l/u'_l)

$$\left(\frac{p'_l}{u'_l}\right) \simeq 1\bigg/\left(\frac{\bar{M}_l}{\bar{c}_l}\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\bigg|_l + \frac{\omega^*}{\bar{c}_l}\right)\left[\frac{\bar{\rho}_l\bar{c}_l\bar{M}_l}{c_p}\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\bigg|_l\left(\frac{s'_l}{u'_l}\right) - \bar{\rho}_l(1+\bar{M}_l)\omega^* - \bar{\rho}_l\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\bigg|_l\right]. \tag{3.28}$$

Upon substitution of (3.28) into (3.25)

$$2\left(\frac{s'_l}{u'_l}\right)\omega^* \simeq \frac{\bar{q}_l R}{\bar{p}_l}\left[\frac{1}{\bar{q}_l}\left(\frac{q'_l}{u'_l}\right) + \frac{1}{\bar{p}_l}\frac{(1+\bar{M}_l)\bar{\rho}_l\omega^* + \bar{\rho}_l\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\Big|_l - (\bar{\rho}_l\bar{c}_l\bar{M}_l/c_p)(s'_l/u'_l)\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\Big|_l}{(\bar{M}_l/\bar{c}_l)\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\Big|_l + \omega^*/\bar{c}_l} - \frac{1}{\bar{u}_l}\right]. \tag{3.29}$$

For a given transfer function q'_l/u'_l, (3.29) can be solved to find ω^*.

Equation (3.29) is substantially simplified for the two cases studied in this paper. For the case where entropy generation is suppressed, the right-hand side of (3.25) vanishes and hence ω^* becomes zero, as expected. For the case with steady heat communication, the first term in the right-hand side of (3.29) is zero and therefore this equation reduces to a quadratic in ω^*. Table 1 presents the values of the corner frequency obtained by solving (3.29) for the steady heat communication case studied in § 4.2.2, as well as the relative error of the approximate results relative to ω^* obtained from the numerical simulations. As expected, the error tends to zero as $\epsilon \to 0$, since the scaling becomes exact in this case. It is noted that the method presented here is not limited to non-reflecting downstream boundary conditions, or the two cases of heat communication studied, as long as the impedance of the downstream boundary and the transfer function of the unsteady heat communication are known.

3.3.3. *Some practical implications of the entropic corner frequency*

This strong dependence of the entropic corner frequency on the particular problem should have practical significance when studying sound propagation and generation in combusting flows. This frequency does not have an upper bound for all flows, and was shown to be exactly zero for one particular form of unsteady heat communication. Thus, since different combustors feature very different distributions of mean and unsteady heat release, as well as very different boundary conditions, a source term of a given magnitude may or may not be generating significant sound, depending on the particular application and its frequency. It is nonetheless intended that the proposed entropic corner frequency may permit a less problematic analysis of the interaction between sound and combusting flows in the higher-frequency range. It remains to be seen how high this frequency may be relative to other dynamics of interest in actual combustors.

4. Summary and conclusions

This paper has presented a comparative analysis of the budgets of acoustic energy and Myers' second-order disturbance energy in a simple inhomogeneous flow with heat communication. The flow considered was non-diffusive and one-dimensional, with excitation by downstream-travelling acoustic and entropic disturbances. Two forms of heat communication were examined: a case with only steady heat communication and another in which unsteady heat addition cancelled the generation of entropy disturbances throughout the inhomogeneous region. The frequency response of the acoustic and disturbance energy budgets were determined analytically for a non-compact inhomogeneous region with zero mean flow. Analytic results for a compact region with mean flow were also presented. Both sets of analytic results were used to validate high-order accurate numerical simulations of the quasi-one-dimensional Euler equations for a non-compact inhomogeneous region with mean flow.

It was shown that for acoustic or entropic excitation of the case with mean flow and steady heat addition, significant entropic disturbances were generated at low frequency, such that entropic terms in the disturbance energy flux dominated the acoustic energy flux. However, regardless of the form of heat communication, the generation of entropic disturbances must diminish with increased forcing frequency and the source terms then produce mainly sound in response to acoustic excitation. The reasons for this trend were first argued on simple scaling grounds.

A method was then introduced for determining an approximate frequency beyond which the generation of entropy disturbances for acoustic excitation could be ignored, and the more general disturbance energy flux approximated the acoustic energy flux. This method accommodated arbitrary forms of heat communication and also upstream and downstream impedances. For the cases studied, it agreed very well with the numerical simulations. However, because it was shown analytically that one particular form of unsteady heat communication can result in this corner frequency being zero, the actual corner frequency for an arbitrary form of heat communication can vary widely and does not have an upper bound for all flows. This is expected to have practical significance when studying sound generation and propagation in combusting flows in particular.

A second result was that sound was shown to be generated by fluid motion experiencing only steady heat communication. For such flows with entropic excitation, sound generation was strongly Mach-number dependent, with the second-order energy analysis becoming singular at sonic flow conditions. Sound generation by steady, heat communication was argued to be similar to the known mechanism of sound generation by acceleration of density disturbances, since both involve source terms containing steady static temperature gradients. This should also be of practical significance for combusting flows, as combustion is completed around the combustor exit.

Appendix A. Analytical analysis for zero mean flow

In this section, a general solution for the acoustic wave propagation in a non-diffusive stationary flow is applied to the problem in figure 1. The appropriate boundary conditions are found and a particular solution is developed. The sound reflection and transmission coefficients are then derived.

Sujith *et al.* (1995) expressed the acoustic wave equation in a varying mean temperature medium as

$$\frac{\mathrm{d}^2 P}{\mathrm{d}x^2} + \frac{1}{\bar{\theta}}\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}x}\frac{\mathrm{d}P}{\mathrm{d}x} + \frac{\omega^2}{\gamma R \bar{\theta}} P = 0, \tag{A 1}$$

in which P is the spatial component of the harmonic pressure fluctuations defined in (3.15). Assuming that the mean temperature $\bar{\theta}$ varies linearly with x then $\bar{\theta} = \bar{\theta}_0 + mx$ where $m = (\bar{\theta}_l - \bar{\theta}_0)/l$ and l is the length of the varying mean temperature region.

A new independent variable κ is introduced as

$$\bar{\theta} = \frac{m^2 \gamma R}{4\omega^2}\kappa^2. \tag{A 2}$$

Substituting this variable into the wave equation (A 1) converts it to a standard Bessel equation of zeroth order, with a general solution of the form

$$P = c_1 J_0(\kappa) + c_2 Y_0(\kappa) = c_1 J_0\left(\frac{\omega}{a}\sqrt{\bar{\theta}}\right) + c_2 Y_0\left(\frac{\omega}{a}\sqrt{\bar{\theta}}\right), \tag{A 3}$$

where c_1 and c_2 are functions of frequency ω only, J_0 and Y_0 are the Bessel and Neumann functions and a is a constant defined as $a = |m|\sqrt{\gamma R}/2$. Through application of the linearized momentum equation, it can also be shown that the acoustic velocity fluctuation is

$$U(x) = -\frac{m}{|m|}\frac{\mathrm{i}}{\bar{\rho}\sqrt{\gamma R \bar{\theta}}}\left[c_1 J_1\left(\frac{\omega}{a}\sqrt{\bar{\theta}}\right) + c_2 Y_1\left(\frac{\omega}{a}\sqrt{\bar{\theta}}\right)\right]. \tag{A 4}$$

The boundary conditions which specify c_1 and c_2 are derived from the configuration of the problem. Assume that the incident acoustic wave I in figure 1 has an associated velocity fluctuation such that $U(0) = \upsilon$, where υ is a constant. Since ideally there is no reflection at the other end of the varying temperature region, it has the same boundary condition as a semi-infinite duct in which the homogeneous wave equation is valid. This semi-infinite region contains only right-travelling waves with

$$p'(x,t) = A \exp\left(\frac{\mathrm{i}\omega x}{\bar{c}_l}\right)\exp(\mathrm{i}\omega t), \tag{A 5}$$

$$u'(x,t) = \left(\frac{A}{\bar{\rho}_l \bar{c}_l}\right)\exp\left(\frac{i\omega x}{\bar{c}_l}\right)\exp(\mathrm{i}\omega t), \tag{A 6}$$

where A is a complex constant, $\bar{\rho}$ is a mean density and $\bar{c}$ is a mean sonic velocity. Since there is no temperature jump at the interface of the semi-infinite duct and the varying temperature region, the acoustic pressure and velocity exhibit

$$p'(l_-,t) = p'(l_+,t), \quad u'(l_-,t) = u'(l_+,t). \tag{A 7a,b}$$

These conditions combined with the solution of the wave equation in the semi-infinite duct give the anechoic boundary condition

$$P(l) = \bar{\rho}_l \bar{c}_l U(l). \tag{A 8}$$

Equation (A 3) then allows determination of the constants c_1 and c_2 in the following form

$$c_1 = \frac{B}{BC - AD}, \qquad c_2 = -\frac{A}{BC - AD}, \tag{A 9a,b}$$

in which A, B, C and D are

$$A = J_0(\beta_l) - \lambda J_1(\beta_l), \quad B = Y_0(\beta_l) - \lambda Y_1(\beta_l), \tag{A 10a,b}$$

$$C = \alpha J_1(\beta_0), \quad D = \alpha Y_1(\beta_0). \tag{A 10c,d}$$

In the above relations α, β_0, β_l and λ are functions of the mean temperature and frequency,

$$\beta_0 = \frac{\omega}{a}\sqrt{\bar{\theta}_0}, \quad \beta_l = \frac{\omega}{a}\sqrt{\bar{\theta}_l}, \quad \alpha = -\frac{m}{|m|}\frac{\mathrm{i}}{\bar{\rho}_0\sqrt{\gamma R \bar{\theta}_0}}, \quad \lambda = -\mathrm{i}\frac{m}{|m|}. \tag{A 11a,d}$$

A.1. *The reflection and transmission coefficients*

Considering the upstream homogeneous region, the pressure and velocity at the origin are

$$P(0) = I + R, \quad U(0) = \frac{1}{\bar{\rho}_0 \bar{c}_0}(I - R), \tag{A 12a,b}$$

where I and R are the amplitudes of the incident and reflected waves, respectively. By employing (A 5) and (A 6) to express the pressure and velocity at the origin, the reflection coefficient based on the wave amplitudes can be written as

$$\frac{R}{I} = \frac{c_1 J_0(\omega/a\sqrt{\bar{\theta}_0}) + c_2 Y_0(\omega/a\sqrt{\bar{\theta}_0}) - \rho_0\sqrt{\gamma R \bar{\theta}_0}}{c_1 J_0(\omega/a\sqrt{\bar{\theta}_0}) + c_2 Y_0(\omega/a\sqrt{\bar{\theta}_0}) + \rho_0\sqrt{\gamma R \bar{\theta}_0}}. \tag{A 13}$$

A similar argument can be made for the transmission coefficient based on wave amplitudes, which leads to

$$\frac{T}{I} = 2\frac{c_1 J_0(\omega/a\sqrt{\bar{\theta}_l}) + c_2 Y_0(\omega/a\sqrt{\bar{\theta}_l})}{c_1 J_0(\omega/a\sqrt{\bar{\theta}_0}) + c_2 Y_0(\omega/a\sqrt{\bar{\theta}_0}) + \rho_0\sqrt{\gamma R \bar{\theta}_0}}. \tag{A 14}$$

Appendix B. Analytical analysis for a compact region with non-zero mean flow

Application of the linearized conservation equations of mass, momentum and either the energy or transport of entropy to a compact inhomogeneous region in figure 1, yields the following expressions for the reflection and transmission coefficients.

B.1. *Acoustic excitation,* ($s_0 = 0, I \neq 0$)

In the case of zero heat unsteadiness

$$\frac{T}{I} = \frac{(a_1 d_2 - a_2 d_1)}{(a_1 b_2 - a_2 b_1)}, \quad \frac{R}{I} = \frac{(b_2 d_1 - b_1 d_2)}{(a_1 b_2 - a_2 b_1)}, \tag{B 1a,b}$$

in which a_1, a_2, b_1, b_2, d_1 and d_1 are

$$a_1 = -\bar{M}_0^2 + 2\bar{M}_0 - \bar{M}_l\frac{\bar{c}_l}{\bar{c}_0} + \bar{M}_0\bar{M}_l\frac{\bar{C}_l}{\bar{C}_0} - 1, \tag{B 2a}$$

$$a_2 = \frac{\bar{c}_0}{\gamma - 1} - \frac{\gamma}{\gamma - 1}\bar{M}_0\bar{c}_0 - \tfrac{3}{2}\bar{M}_0^2\bar{c}_0 + \tfrac{1}{2}\bar{M}_l^2\frac{\bar{c}_l^2}{\bar{c}_0} - \tfrac{1}{2}\bar{M}_0\bar{M}_l^2\frac{\bar{c}_l^2}{\bar{c}_0} + \tfrac{1}{2}\bar{M}_0^3\bar{c}_0, \tag{B 2b}$$

$$b_1 = 1 + \frac{\bar{\rho}_0}{\bar{\rho}_l}\frac{\bar{c}_0}{\bar{c}_l}\bar{M}_0, \tag{B 2c}$$

$$b_2 = \frac{\bar{c}_l}{\gamma - 1} + \frac{\gamma}{\gamma - 1}\bar{M}_l\bar{c}_l + \frac{\bar{\rho}_0}{\bar{\rho}_l}\bar{M}_0\bar{M}_l\bar{c}_0, \tag{B 2d}$$

$$d_1 = 1 + 2\bar{M}_0 - \bar{M}_l\frac{\bar{c}_l}{\bar{c}_0} - \bar{M}_0\bar{M}_l\frac{\bar{c}_l}{\bar{c}_0} + \bar{M}_0^2, \tag{B 2e}$$

$$d_2 = \frac{\bar{c}_0}{\gamma - 1} + \frac{\gamma}{\gamma - 1}\bar{M}_0\bar{c}_0 - \tfrac{3}{2}\bar{M}_0^2\bar{c}_0 + \tfrac{1}{2}\bar{M}_l^2\frac{\bar{c}_l^2}{\bar{c}_0} - \tfrac{1}{2}\bar{M}_0^3\bar{c}_0 + \tfrac{1}{2}\bar{M}_0\bar{M}_l^2\frac{\bar{c}_l^2}{\bar{c}_0}. \tag{B 2f}$$

For the second form of heat unsteadiness such that $Ds'/Dt = 0$, similar analysis gives

$$\frac{R}{I} = \frac{\dfrac{1 + \bar{M}_l}{1 + \bar{M}_0} - \dfrac{\bar{c}_0}{\bar{c}_l}}{\dfrac{\left(1 - \bar{M}_0\right)\left(1 + \bar{M}_l\right)}{(1 + M_0)^2} + \dfrac{\bar{c}_0}{\bar{c}_l}\dfrac{(1 - M_0)^2}{(1 + M_l)^2}}, \tag{B 3a}$$

and

$$\frac{T}{I} = \frac{\dfrac{2}{1 + \bar{M}_0}}{\dfrac{\bar{c}_0}{\bar{c}_l}\dfrac{(1 + \bar{M}_l)(1 - \bar{M}_0)}{(1 + \bar{M}_l)^2} + \left(\dfrac{1 + \bar{M}_l}{1 + \bar{M}_0}\right)^2}. \tag{B 3b}$$

B.2. *Entropic excitation,* $(I = 0, s_0 \neq 0)$

In the case where $q' = 0$

$$\frac{R}{s_0} = \frac{(h_1 b_2 - b_1 h_2)(f_1 e_2 - d_2 h_1) + (e_1 h_2 - h_1 e_2)(b_2 f_2 - a_2 h_2)}{(f_1 b_2 - a_1 h_2)(f_1 e_2 - d_2 h_1) + (d_1 h_2 - f_1 e_2)(b_2 f_2 - a_2 h_2)}, \tag{B 4a}$$

$$\frac{T}{s_0} = \frac{(h_1 b_2 - b_1 h_2)(f_1 e_2 - d_1 h_2) + (e_1 h_2 - h_1 e_2)(f_1 b_2 - a_1 h_2)}{(f_2 b_2 - a_2 h_2)(f_1 e_2 - d_1 h_2) + (d_2 h_2 - f_2 e_2)(f_1 b_2 - a_1 h_2)}, \tag{B 4b}$$

where

$$a_1 = \frac{1}{\bar{c}_0}(\bar{M}_0 - 1), \quad a_2 = \frac{1}{\bar{c}_l}(\bar{M}_l + 1), \tag{B 5a, b}$$

$$b_1 = -\frac{\bar{\rho}_0\bar{M}_0\bar{c}_0}{C_p}, \quad b_2 = -\frac{\bar{\rho}_l\bar{M}_l\bar{c}_l}{C_p}, \tag{B 5c, d}$$

$$d_1 = \frac{C_p\bar{\theta}_0}{\bar{c}_0}(M_0 - 1) + \bar{M}_0\bar{c}_0(1 - \bar{M}_0), \tag{B 5e}$$

$$d_2 = \frac{C_p\bar{\theta}_l}{\bar{c}_l}(M_l + 1) + \bar{M}_l\bar{c}_l(1 + \bar{M}_l), \tag{B 5f}$$

$$e_1 = \frac{\bar{\rho}_0\bar{M}_0\bar{c}_0^3}{(\gamma - 1)C_p} - \bar{\rho}_0\bar{M}_0\bar{\theta}_0\bar{c}_0, \quad e_2 = \frac{\bar{\rho}_l\bar{M}_l\bar{c}_l^3}{(\gamma - 1)C_p} - \bar{\rho}_l\bar{M}_l\bar{\theta}_l\bar{c}_l, \tag{B 5g, h}$$

$$f_1 = \bar{M}_0^2 - 2\bar{M}_0 + 1, \quad f_2 = \bar{M}_l^2 + 2\bar{M}_l + 1, \tag{B 5i, j}$$

$$h_1 = -\frac{\bar{\rho}_0 \bar{M}_0^2 \bar{c}_0^2}{C_p}, \quad h_2 = -\frac{\bar{\rho}_l \bar{M}_l^2 \bar{c}_l^2}{C_p}. \tag{B 5k, l}$$

For the second type of heat unsteadiness such that $Ds'/Dt = 0$

$$\frac{R}{s_1} = \frac{da_2}{a_2 b_1 - a_1 b_2}, \quad \frac{T}{s_1} = \frac{da_1}{a_1 b_2 - a_2 b_1}, \tag{B 6a, b}$$

in which

$$a_1 = \frac{-1}{\bar{c}_0}(1 - \bar{M}_0), \quad a_2 = \frac{-1}{\bar{c}_l}(1 + \bar{M}_l), \tag{B.7a, b}$$

$$b_1 = (1 - \bar{M}_0)^2, \quad b_2 = -(1 + \bar{M}_l)^2, \tag{B.7c, d}$$

$$d = \frac{1}{C_p}\bar{\rho}_0 \bar{M}_0 \bar{c}_0^2 \left(\bar{M}_0 - \bar{M}_l \frac{\bar{c}_l}{\bar{c}_0} \right). \tag{B.7e}$$

In the above relations the downstream mean Mach number $\bar{M}_l$ is a function of $\bar{M}_0$ as well as the upstream mean temperature, and is found by iterative solution e.g. Oates (1984),

$$\bar{M}_l^2 = \frac{2f}{1 - 2\gamma f + [1 - 2(\gamma + 1) f]^{1/2}}, \tag{B.8a}$$

$$f = \frac{1 + [(\gamma - 1)/2]\bar{M}_0^2}{\left(1 + \gamma \bar{M}_0^2\right)^2} \left(\frac{\bar{\theta}_{t0}}{\bar{\theta}_{tl}} \right) \bar{M}_0^2, \tag{B.8b}$$

where $\bar{\theta}_t$ is the mean stagnation temperature.

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